Gauge-Invariant Differential Renormalization: Abelian Case[1]

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Abstract

A new version of differential renormalization is presented. It is based on pulling out certain differential operators and introducing a logarithmic dependence into diagrams. It can be defined either in coordinate or momentum space, the latter being more flexible for treating tadpoles and diagrams where insertion of counterterms generates tadpoles. Within this version, gauge invariance is automatically preserved to all orders in Abelian case. Since differential renormalization is a strictly four-dimensional renormalization scheme it looks preferable for application in each situation when dimensional renormalization meets difficulties, especially, in theories with chiral and super symmetries. The calculation of the ABJ triangle anomaly is given as an example to demonstrate simplicity of calculations within the presented version of differential renormalization.

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1 Introduction

Various versions of differential renormalization [1–7] explicitly describe how a product of propagators corresponding to a given graph and considered in coordinate space can be defined as a distribution on the whole space of test functions, starting from the subspace of test functions which vanish in a vicinity of points where the coordinates coincide. Differential renormalization provides a strictly four-dimensional renormalization scheme useful for applications [8, 9]. It is based on 'pulling out' some differential operators from the initial unrenormalized diagram; these are either $\Box = \partial_\alpha \partial_\alpha$ [1–4,7], or $\hat{S} = \frac{1}{2}\frac{\partial}{\partial x_\alpha} x_\alpha$ [5, 6]. The latter version looks more natural for generalization to the case with non-zero masses when one uses the operator

$$\hat{S} = \frac{1}{2}\sum_i \frac{\partial}{\partial u_i} u_i - \frac{1}{2}\sum_l m_l \frac{\partial}{\partial m_l}, \tag{1}$$

where u_i are independent coordinate differences.

The purpose of this paper is to present a new version of differential renormalization which automatically preserves gauge symmetry at least in the Abelian case. To do this we shall modify, in the next section, the version of ref. [5, 6] based on operator (1). The renormalization is defined recursively, for every graph, provided it was defined for graphs and reduced graphs with a smaller number of loops. It reduces to pulling out the operator $\hat{S}$ and inserting some logarithm into the given Feynman amplitude. The main idea of the new version of differential renormalization is to 'spoil' the diagram by the logarithms in a minimal way, in the sense that a minimal number of propagators is spoiled by the logarithms.

In Section 3 we shall present a gauge-invariant differential renormalization scheme for QED. The underlying idea is simple: to insert the logarithms into the photon lines. This enables us to perform, for renormalized Feynman integrals, the same manipulations for establishing the gauge invariance as for unrenormalized ones. Thus gauge invariance is maintained automatically, to all orders.[3] One-loop polarization of vacuum requires special treatment when an operator with explicitly transverse structure is pulled out before renormalization.

In Section 4, by translating renormalization prescription into the language of momentum space, we shall present general recipes applicable to arbitrary diagrams. In particular, we shall treat, in a more simple way (as compared to ref. [6]), renormalization of tadpoles and diagrams where insertion of counterterms for subgraphs generates tadpoles (e.g. the sun-set diagram). Furthermore, the calculation of the triangle ABJ anomaly is presented as an example of the application of differential renormalization.

Finally, Section 5 contains discussion of the obtained results.

[3]In [10] a recipe of differential renormalization formulated in momentum space and applicable for diagrams with simple topology of divergences was applied to prove relations relevant to Ward identities and corresponding to some partial classes of diagrams that contribute to Green functions.

2 Differential renormalization in coordinate space

2.1 Renormalization prescriptions

A Feynman amplitude is defined in coordinate space through products of propagators

$$\Pi_\Gamma(x_1, \ldots, x_N) \equiv \Pi_\Gamma(\underline{x}) = \prod_l \tfrac{1}{i} G_l(x_{\pi_+(l)} - x_{\pi_-(l)}), \tag{2}$$

taken over lines of a given graph Γ. Here $\pi_\pm(l)$ are respectively beginning and the end of a line l, and

$$G_l(x) = \frac{im_l}{4\pi^2}\, P_l(-i\partial/\partial x, m)\, \frac{K_1(m_l\sqrt{-x^2+i0})}{\sqrt{-x^2+i0}} \equiv P_l(-i\partial/\partial x, m) G(x) \tag{3}$$

is a propagator, with P_l polynomial and K_1 a modified Bessel function.

The strategy of differential renormalization is an explicit realization of the extension of functional (2), from a subspace of test functions to the whole space (see details in refs. [5, 6]). Instead of renormalization prescriptions of [5, 6], let us now define renormalization of Feynman amplitudes by the following recursive procedure. Note that we need to renormalize not only Feynman amplitudes themselves but as well Feynman amplitudes $\Pi_\Gamma^{j,l}(\underline{x})$ 'spoiled' in some way by additional logarithmical dependence. This notation implies that the j-th power of some logarithm is introduced in a certain way into some fixed l-th line:

$$\Pi_\Gamma^{j,l}(\underline{x}) = \tfrac{1}{i} G_l^{(j)}(x_{\pi_+(l)} - x_{\pi_-(l)}) \prod_{l' \neq l} \tfrac{1}{i} G_l(x_{\pi_+(l')} - x_{\pi_-(l')}). \tag{4}$$

For example, one can substitute $G_l(x)$ (corresponding to this very line) by

$$G_l^{(j)}(x) = \ln^j \mu^2 x^2 G_l(x). \tag{5}$$

However this is not the simplest version suitable for applications. A better variant is based on multiplication by a logarithm in momentum space:

$$G_l^{(j)}(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 q\, (-1)^j \ln^j \left(H(q)/\mu^2\right) \tilde{G}_l(q), \tag{6}$$

where the function H can be chosen as $-q^2 - i0$ or $m_l^2 - q^2 - i0$. The latter version generally looks much more simple from the point of view of the calculational simplicity. In all these situations, the logarithmically spoiled propagator satisfies

$$\hat{D} G_l^{(j)}(x) = j G_l^{(j-1)}(x) - (1 + a_l) G_l^{(j)}(x) \tag{7}$$

with

$$\hat{D} = \frac{1}{2} x \frac{\partial}{\partial x} - \frac{1}{2} m_l \frac{\partial}{\partial m_l} \equiv \hat{S} - 2 \tag{8}$$

and a_l degree of the polynomial P_l in (3).

Suppose that we know how to renormalize all the Feynman amplitudes $\Pi_\gamma^{j',l}(\underline{x})$ and $\Pi_{\Gamma/\gamma}^{j',l}(\underline{x})$ corresponding to reduced graphs Γ/γ and proper subgraphs $\gamma \subset \Gamma$ of the given graph Γ. (These subgraphs and reduced graphs have smaller number of loops, h. Recall that the reduced graph Γ/γ is obtained from Γ by reducing each connectivity component of γ to a point.) Here l is a fixed line of Γ, and j' is arbitrary integer. This means that we know all counterterms contributing to the R-operation (i.e. renormalization at the diagrammatical level) which acts on the corresponding Feynman amplitudes:

$$\begin{aligned} R\Pi_\gamma^{j',l} &= \sum_{\gamma_1,\dots,\gamma_j} \Delta(\gamma_1)\dots\Delta(\gamma_j)\Pi_\gamma^{j',l} \\ &\equiv R'\Pi_\gamma^{j',l} + \Delta(\gamma)\Pi_\gamma^{j',l}. \end{aligned} \tag{9}$$

where γ stands either for a subgraph or reduced graph, Δ is the corresponding counterterm operation, and the sum is over all sets $\{\gamma_1,\dots,\gamma_j\}$ of disjoint divergent 1PI subgraphs of γ, with $\Delta(\emptyset) = 1$. The operation R' is called incomplete R-operation.

In this and the following section we shall consider diagrams without tadpoles and such that insertion of counterterms for subgraphs does not generate tadpoles. Note that this is sufficient for QED.

Let us now define renormalization of the Feynman amplitude $\Pi_\Gamma^{j,l}$ as:

$$R\,\Pi_\Gamma^{j,l} = \frac{1}{j+1}\left(\hat{S}+\omega/2\right)\left(1-M^{\omega-1}\right)\,R'\,\Pi_\Gamma^{j+1,l} - \frac{1}{j+1}\sum_{\gamma\subset\Gamma:\, l\overline{\in}\gamma} R\,\mathcal{C}_\gamma\Pi_\Gamma^{j+1,l}, \tag{10}$$

Here R' is incomplete R-operation given by (9), and $\omega = 4h - 2L + a$ is the degree of divergence (with L the number of lines and a total degree of the polynomials P_l in (3)). The sum in the second term of the right-hand side of (10) runs over all 1PI proper subgraphs γ of Γ that do not include the line l. The operator M^r performs Taylor expansion of order r in masses and external momenta of the graph Γ.

Furthermore,

$$\Delta(\Gamma)\Pi_\Gamma^{j,l} = R\Pi_\Gamma^{j,l} - R'\,\Pi_\Gamma^{j,l}. \tag{11}$$

Finally, the operations $\mathcal{C}_\gamma$ are determined from equations

$$\left(\hat{S}+\omega/2\right)R\,\Pi_\Gamma = \sum_{\gamma\subseteq\Gamma}\mathcal{C}_\gamma R\Pi_\Gamma \equiv \sum_{\gamma\subset\Gamma} R\mathcal{C}_\gamma\Pi_\Gamma + \mathcal{C}_\Gamma\Pi_\Gamma, \tag{12}$$

where the operation $\mathcal{C}_\gamma$ inserts a polynomial $\mathcal{P}_\gamma$ of degree $\omega(\gamma)$ in masses of γ and its external momenta into the reduced vertex of the graph Γ/γ. Symbolically we write

$$\mathcal{C}_\gamma\Pi_\Gamma = \Pi_{\Gamma/\gamma}\circ\mathcal{P}_\gamma \tag{13}$$

where $\circ$ denotes the insertion operation. In the language of coordinate space,

$$\mathcal{C}_\Gamma\Pi_\Gamma(\underline{x}) = \mathcal{P}_\Gamma(\partial/\partial x_i)\prod_{i\in\Gamma,i\neq i_0}\delta^{(4)}(x_i - x_{i_0}), \tag{14}$$

where i_0 is a fixed vertex. The polynomial for the graph $\mathcal{P}_\Gamma$ is expressed from the difference of the left-hand side of (12) and the sum in the right-hand side.

2.2 Comments and examples

1. A proof of the fact that relations (10)–(12) define a correct R-operation can be obtained by a trivial modification of the arguments presented in ref. [6]. In particular, it is proved that in the coordinate space with deleted origin (i.e. when at least one of the coordinate difference variables $x_i - x_j$ is non-zero) the incompletely renormalized Feynman amplitude $R'\,\Pi_\Gamma^{j,l}$ can be written as the right-hand side of eq. (10). Then this equation enables us to extend the initial functional to the whole space of test functions so that, by definition, the right-hand side of eq. (10) determines the renormalized quantity $R\,\Pi_\Gamma^{j,l}$. Indeed, the first term there does not have divergences because all subdivergences are removed by R'. As to the overall divergence, it is removed by the product of the operators $\left(\hat{S}+\omega/2\right)(1-M^{\omega-1})$. The preliminary Taylor subtractions result in producing a polynomial in momenta and masses of degree ω and thereby reduce the degree of divergence from ω to zero. After commutation of the operator $\left(\hat{S}+\omega/2\right)$ with this polynomial, the resulting operator $\hat{S}$ acts as in logarithmically divergent case (see details in [5, 6]) by removing the divergence by multiplication of the first degree monomials contained in operator (1).

The other terms in the right-hand side of eq. (10) are manifestly renormalized Feynman amplitudes corresponding to reduced subgraphs (with smaller numbers of loops).

2. Examples. (i) If a diagram is primitively divergent (i.e. does not involve subdivergences), with degree of divergence ω, then the above prescriptions give

$$R\Pi_\Gamma = \left(\hat{S}+\omega/2\right)\left(1-M^{\omega-1}\right)\Pi_\Gamma^{1,l}, \tag{15}$$

where l is an arbitrary line of Γ. Instead of $\Pi_\Gamma^{1,l}$ one can also use a linear combination $\sum \xi_l \Pi_\Gamma^{1,l}$, with $\sum \xi_l = 1$ (see e.g. Sect. 4.2 where the triangle anomaly is calculated).

(ii) To spoil the minimal number of the propagators involved it is natural to choose, as the line l, a line that is chosen for renormalization of some maximal subgraph of the given graph. Let Γ be logarithmically divergent and let it involve only one proper divergent subgraph γ, with $\omega(\gamma)=0$. Then, using some $l \in \gamma$, one has

$$R\Pi_\Gamma = \frac{1}{2}\hat{S}\left(\Pi_{\Gamma\backslash\gamma}\hat{S}\Pi_\Gamma^{2,l}\right). \tag{16}$$

A generalization of this formula for the case when all the divergent subgraphs of the diagram form a nest $\gamma_1 \subset \gamma_2 \subset \ldots \subset \gamma_r \equiv \Gamma$, with $\omega(\gamma_i)=0$, and the initial Feynman amplitude itself involves the j-th power of the logarithm, looks like

$$R\Pi_\Gamma = \frac{(j-1)!}{(j+r)!}\hat{S}\left(\Pi_{\Gamma\backslash\gamma_{r-1}}\hat{S}\left(\Pi_{\Gamma\backslash\gamma_{r-2}}\ldots\hat{S}\Pi_\Gamma^{j+r,l}\right)\right), \tag{17}$$

where $l \in \gamma_1$.

(iii) Let Γ be logarithmically divergent and let it involve only two disjoint divergent subgraphs with $\omega(\gamma_i) = 0$. Let $l_i \in \gamma_i,\ i = 1, 2$. Then one can define

$$R\Pi_\Gamma = \hat{S}\Pi_{\Gamma\setminus(\gamma_1\cup\gamma_2)} \left(\frac{1}{2}\hat{S}\Pi^{2,l_1}_{\gamma_1}\right) \left(\hat{S}\Pi^{1,l_2}_{\gamma_2}\right) - R\left(\mathcal{C}_{\gamma_2}\Pi^{1,l_1}_{\Gamma}\right), \tag{18}$$

where the operation $\mathcal{C}_{\gamma_2}$ is defined by

$$\mathcal{C}_{\gamma_2}\Pi_{\gamma_2} = \hat{S}\Pi_{\gamma_2} \equiv c_{\gamma_2} \prod_{i\in\Gamma, i\neq i_0} \delta^{(4)}(x_i - x_{i_0}).$$

(iv) Let Γ possess the following family of logarithmically divergent subgraphs: $\gamma_1, \gamma_2, \gamma_{12} \equiv \gamma_1 \cap \gamma_2$ and Γ itself. Let $l \in \gamma_{12}$. Then

$$R\Pi_\Gamma = \hat{S}R'\Pi^{1,l}_\Gamma, \quad R' = 1 + \Delta(\gamma_1) + \Delta(\gamma_2) + \Delta(\gamma_{12}). \tag{19}$$

Here $\Delta(\gamma_i) = R(\gamma_i) - R'(\gamma_i), \Delta(\gamma_{12}) = R(\gamma_{12}) - 1$ are found from (16) and (18) for $j = 1$.

(v) Let Γ be as in the previous example and $\gamma_{12} \equiv l$ (which is not of course divergent). Let us keep in mind the 2-loop photon exchange diagram of QED contributing to the vacuum polarization. We have $\omega(\Gamma) = 2$ and $\omega(\gamma_i) = 1$ for one-loop vertex subgraphs. Then

$$R\Pi_\Gamma = (\hat{S} + 1)(1 - M^1)R'\Pi^{1,l}_\Gamma, \tag{20}$$

$$R' = 1 + \Delta(\gamma_1) + \Delta(\gamma_2), \quad \Delta(\gamma_i) = R(\gamma_i) - 1, \tag{21}$$

$$R\Pi_{\gamma_i} = (\hat{S} + 1/2)(1 - M^0)\Pi^{1,l}_{\gamma_i}, \tag{22}$$

$$R\Pi^{1,l}_{\gamma_i} = \frac{1}{2}(\hat{S} + 1/2)(1 - M^0)\Pi^{2,l}_{\gamma_i}. \tag{23}$$

3. We use here the operator $\left(\hat{S} + \omega/2\right)(1 - M^{\omega-1})$ instead of a differential operator of order $\omega + 1$ of the previous version [5, 6]. Although this mixture of coordinate and momentum space operators might seem strange we insist that this version of differential renormalization is more simple that previous ones and this preliminary momentum subtraction is natural. In Section 4 this version will be translated into momentum space which happens to be appropriate for treating arbitrary diagrams. To apply the operator $(1 - M^{\omega-1})$ one can turn to momentum space, perform Taylor expansion and then come back to coordinate space. However this momentum subtraction is as well easily described in coordinate space. In particular, in the massless case, we have the following prescription for the action of the functional $(1 - M^{\omega-1}_q)\Pi(x)$ on a test function $\phi(x)$:

$$\left((1 - M^{\omega-1}_q)\Pi(x),\ \phi(x)\right) = \int \mathrm{d}x\Pi(x)(1 - M^{\omega-1}_x)\phi(x), \tag{24}$$

where $M_x^{\omega-1}$ performs Taylor expansion in x.

4. It is also possible to 'logarithmically spoil' the diagrams in the language of the α-representation, by inserting factors like $\ln^j \mu^{2h} D(\underline{\alpha})$ into the integrand of the α-representation (here $D(\underline{\alpha})$ is a standard homogeneous function). However this possibility looks disadvantageous because of the lack of control of manipulations that are relevant for establishing desired symmetry. Note that momentum space modification (6) of the propagator is easily described in the α-representation as insertion of $\ln \mu'^2 \alpha_l$ (with μ' proportional to μ in (6)).

5. For the products of propagators in coordinate space all the vertex are considered as external. Feynman amplitudes are obtained from the products Π_Γ by integrating over coordinates associated with internal vertices:

$$F_\Gamma(x_1, \ldots, x_n) = \int \mathrm{d}x_{n+1} \ldots \mathrm{d}x_N \Pi_\Gamma(x_1, \ldots, x_N). \tag{25}$$

When writing down renormalization prescriptions (10)–(12) for Feynman amplitudes (25) one obtains similar formulae where the operator $\hat{S}$ is now given by the sum over external coordinates.

6. The coefficients polynomials $\mathcal{P}_\gamma$ play the role of contributions of simple poles to counterterms within dimensional renormalization. They are related with renormalization group functions by the same formulae as in the case of the previous version of differential renormalization — see [6].

3 Gauge-invariant differential renormalization of QED

As is well-known the Ward identities in QED, e.g.

$$q_\alpha \Gamma^\alpha(p, q, p+q) = \Sigma(p+q) - \Sigma(p), \tag{26}$$

that connects the vertex and self-energy Green functions are proved using standard manipulations based on the following identity involving the electron propagator:

$$\frac{\partial}{\partial \xi_\alpha} [S(x_2 - \xi)\gamma_\alpha S(\xi - x_1)] = \frac{1}{i}[\delta(x_2 - \xi) - \delta(\xi - x_1)]S(x_2 - x_1), \tag{27}$$

which are based on the equation of motion for the free electron propagator $S(x) \equiv (m + i \not{\partial})G(x)$, namely $(m - i \not{\partial})S(x) = \delta(x)$.

In fact one uses (see, e.g., [11]) a natural one-to-one correspondence between diagrams that contribute to these Green functions (vertex diagrams are obtained by insertion a new triple vertex into lines of the electron path between external electron lines). The main problem in establishing the Ward identity is to prove that these manipulations are also valid for the *renormalized* Feynman amplitudes. Let

us now use the structure of our renormalization procedure which reduces to pulling out differential operators and spoiling the Feynman amplitudes by logarithms. Note that commutation relations of the differential operators with monomials in external momenta (i.e. derivatives in coordinates) are very simple:

$$P^r \hat{S} = (\hat{S} + r/2)(1 - M^{r-1})P^r, \tag{28}$$

where P^r is such a monomial of degree r (the value $r = 1$ is relevant for (26)).

Therefore, the problem is not to spoil identities (27) by the inserted logarithms. Since these identities are connected with electron lines a natural and simple solution of this problem is to introduce the logarithms only into photon lines. Then the proof of (26) is performed by induction (as the renormalization prescriptions themselves), with the use of (10), under assumption that the corresponding relation between vertex and self-energy diagrams is valid for all subgraphs and reduced graphs. A crucial point is that in the right hand side of (10) one has reduced graphs with at least one photon line if the initial graph has a photon line, because summation in the right-hand side is over subgraphs γ such that $l\overline{\in}\gamma$.

The Ward identities that connect N-point and $N-1$-point Green functions, with $N > 3$ and $N = 2$, are analogously proved. Only diagrams without photon lines require special treatment. These are the one-loop polarization of vacuum, vertex and box diagram. However, the second one is zero, the third is convergent. Thus, to complete consideration of the case $N = 2$ it is sufficient to prove that the one-loop polarization of vacuum is transverse. A straightforward application of general prescription (10) leads to a non-transverse expression. Of course, it is possible to adjust finite counterterms and arrive at a gauge-invariant result. (Why is it bad to do this just for *one diagram* of the theory?) However, following the style of differential renormalization, one can modify the corresponding prescriptions by pulling out differential operators in a manifestly gauge-invariant way. Using manipulations [7] based on standard properties of Bessel functions, the one-loop polarization of vacuum can be written for $x \neq 0$ as

$$\begin{aligned}\Pi_{\mu\nu}(x) \equiv -e^2 \left(\frac{im^2}{4\pi^2}\right)^2 \mathrm{tr}\left\{(m + i \not{\partial})f(z)\gamma_\mu(m - i \not{\partial})f(z)\right\} \\ = \frac{e^2 m^4}{12\pi^4}(\partial_\mu\partial_\nu - g_{\mu\nu})h(z),\end{aligned} \tag{29}$$

where

$$f(z) = K_1(z)/z, \tag{30}$$

$$h(z) = K_1(z)/z^2 + K_0(z)K_1(z)/z + K_0^2(z) - K_1^2(z), \tag{31}$$

$z = m\sqrt{-x^2 + i0}$, and $K_{0,1}$ modified Bessel functions.

Since the transverse structure is already explicit one can remove divergences in the right-hand side of (29) in an arbitrary way. For example, it is possible to continue

to pull out laplacians as it was down in ref. [7]. However only the first term in the right-hand side of (31) is UV-divergent (i.e. non-integrable in the vicinity of the point $x = 0$). In fact, it is proportional to the one-loop scalar diagram so that one can also apply prescription (15) with $\omega = 0$. Note that all four terms in the right-hand side of (31) admit a natural diagrammatical interpretation and have simple expressions in momentum space, because $K_0(z)$ is obtained from K_1/z (i.e. scalar propagator) by the operator $-m\frac{\mathrm{d}}{\mathrm{d}m}$, and K_1 just by multiplication by m^2x^2 (i.e. $-m^2\Box_q$ in momentum space).

To conclude the section we note that in other theories with Abelian gauge symmetry the situation is quite similar, with unessential additional considerations. For example, in scalar electrodynamics, one should take into account massive tadpoles (which are zero in QED). The tadpoles are not still covered by prescriptions (10)–(12). However, in the next section, we shall arrive at general prescriptions, using the momentum space language.

4 Differential renormalization in momentum space

4.1 General renormalization prescriptions

Let us now turn to momentum space where basic physical quantities are calculated. First we observe that renormalization prescriptions (10)–(12) are trivially transformed into momentum space. (We now distinguish external and internal vertices — see comment 5 in the end of Sect. 2.) Let $F_\Gamma(\underline{q},\underline{m}) \equiv F_\Gamma(q_1,\ldots,q_n,m_1,\ldots,m_L)$ be the Feynman integral given by

$$(2\pi)^4\delta\left(\sum q_j\right)F_\Gamma(\underline{q},\underline{m}) = \int \mathrm{d}x_1\ldots\mathrm{d}x_n \exp\left(i\sum q_j x_j\right)F_\Gamma(\underline{x},\underline{m}), \tag{32}$$

$$F_\Gamma(\underline{q},\underline{m}) = \int \mathrm{d}k_1\ldots\mathrm{d}k_h \tilde{\Pi}_\Gamma(\underline{q},\underline{m}), \tag{33}$$

where $\underline{k} \equiv k_1,\ldots k_h$ is a set of loop momenta of Γ, and $\tilde{\Pi}_\Gamma$ is the product of propagators in momentum space $\tilde{G}_l(q) = P_l(q,m)/(m_l^2 - q^2 - i0)$ associated with the given graph. Then we get the following prescriptions:

$$R\,F_\Gamma^{j,l} = \frac{1}{j+1}\left(\omega/2 - \tilde{D}\right)\left(1 - M^{\omega-1}\right)\,R'\,F_\Gamma^{j+1,l} - \frac{1}{j+1}\sum_{\gamma\subset\Gamma:\, l\overline{\in}\gamma} R\,\mathcal{C}_\gamma F_\Gamma^{j+1,l}, \tag{34}$$

$$\Delta(\Gamma)F_\Gamma^{j,l} = RF_\Gamma^{j,l} - R'\,F_\Gamma^{j,l}, \tag{35}$$

$$\left(\omega/2 - \tilde{D}\right)R\,F_\Gamma = \sum_{\gamma\subseteq\Gamma}\mathcal{C}_\gamma R\Pi_\Gamma \equiv \sum_{\gamma\subset\Gamma} R\mathcal{C}_\gamma F_\Gamma + \mathcal{C}_\Gamma F_\Gamma. \tag{36}$$

Thus the only distinction is that instead of operator $\hat{S}$ we now have dilatation

operator (times 1/2)

$$\tilde{D} = \frac{1}{2}\sum_i q_i \frac{\partial}{\partial q_i} + \frac{1}{2}\sum_l m_l \frac{\partial}{\partial m_l}. \tag{37}$$

By definition it acts now on integrands of Feynman integrals *before integration* in loop momenta.

For example, for one-loop scalar Feynman integral with general masses we have

$$R \int \mathrm{d}k \frac{\ln^j (m_1^2 - k^2 - i0)/\mu^2}{(m_1^2 - k^2 - i0)(m_2^2 - (q-k)^2 - i0)}$$
$$= \frac{1}{j+1}\int \mathrm{d}k \frac{1}{2}\left(q\frac{\partial}{\partial q} + m_1 \frac{\partial}{\partial m_1} + m_2 \frac{\partial}{\partial m_2}\right) \frac{\ln^{j+1}(m_1^2 - k^2 - i0)/\mu^2}{(m_1^2 - k^2 - i0)(m_2^2 - (q-k)^2 - i0)}. \tag{38}$$

In particular, for $m_1 = m_2 = 0$ and $j = 0$, this reduces to

$$R \int \frac{\mathrm{d}k}{(-k^2 - i0)(-(q-k)^2 - i0)} = -\int \mathrm{d}k \frac{q(q-k)\ln(-k^2 - i0)/\mu^2}{(-k^2 - i0)(-(q-k)^2 - i0)^2}. \tag{39}$$

To calculate (39) it is better not to differentiate the integrand explicitly by operator $\frac{1}{2}q\frac{\partial}{\partial q}$. Rather, it is worthwhile to introduce analytic regularization [13]:

$$RF(q) = \int \mathrm{d}^4 k \left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}\right) \frac{1}{2} q \frac{\partial}{\partial q} \frac{\mu^{2\lambda}}{(-k^2 - i0)^{1+\lambda}(-(q-k)^2 - i0)}\Bigg|_{\lambda=0}. \tag{40}$$

When $\lambda \neq 0$, we may use the following order: to calculate the integral, differentiate in q, differentiate in λ and finally put $\lambda = 0$. When calculating the integral one uses the four-dimensional one-loop formula

$$\int \mathrm{d}^4 k \frac{1}{(-k^2 - i0)^{\lambda_1}(-(q-k)^2 - i0)^{\lambda_2}} = i\pi^2 G(\lambda_1, \lambda_2) \frac{1}{(-q^2 - i0)^{\lambda_1 + \lambda_2 - 2}}, \tag{41}$$

where G is the four-dimensional G-function

$$G(\lambda_1, \lambda_2) = \frac{\Gamma(\lambda_1 + \lambda_2 - 2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \frac{\Gamma(2-\lambda_1)\Gamma(2-\lambda_2)}{\Gamma(4 - \lambda_1 - \lambda_2)}. \tag{42}$$

In particular,

$$G(1, 1+\lambda) = \frac{1}{\lambda(1-\lambda)}. \tag{43}$$

Finally we have

$$R\int \mathrm{d}^4 k \frac{1}{(-k^2 - i0)((q-k)^2 - i0)} = i\pi^2 \left(1 - \ln(-q^2 - i0)/\mu^2\right). \tag{44}$$

For example 2 of Sect. 2 with a nest of divergent subgraphs, as well as for other examples, momentum space versions are quite similar, with product of propagators in

the Feynman integral in momentum space and operators $\tilde{D}$ (instead of $-\hat{S}$) associated with external momenta and internal masses of corresponding subgraphs.

Note that the arguments used to prove the coordinate space prescriptions can be also translated in momentum space language.[4] In fact, one starts with an incompletely renormalized Feynman integral and writes down the formula of (strictly four-dimensional) integration by parts. Then the coordinate space procedure of extension of the product of distributions to the whole space looks, in momentum space, as dropping surface terms in this formula (which are polynomials in external momenta and correspond to counterterms). To resolve the structure of (generally overlapping) divergences one here uses sectors in momentum space.

Let us now remember that we considered our prescriptions (10)–(12) and their momentum space versions (34)–(36) only for graphs that do not contain tadpoles and that do not lead to tadpoles when inserting associated counterterms. It turns out that the momentum space prescriptions already have a desired form that is applicable for general graphs. In particular, for the tadpole graph shown in Fig. 1a, the recipe

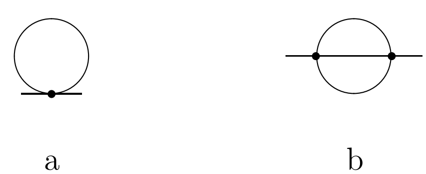

Figure 1: (a) tadpole; (b) sun-set diagram

(34) (that is obtained using the above arguments based on integration by parts in momentum space) gives

$$
\begin{aligned}
R \int \frac{\mathrm{d}k}{m^2 - k^2 - i0} &= \int \mathrm{d}k \left(1 - \frac{1}{2} m \frac{\partial}{\partial m}\right) (1 - M_m^1) \frac{\ln(m^2 - k^2 - i0)/\mu^2}{m^2 - k^2 - i0} \\
&= \int \frac{\mathrm{d}k}{(m^2 - k^2 - i0)^2} \Big\{ [m^2 - k^2 \ln(1 + m^2/(-k^2 - i0))] \\
&\quad -2m^2 \ln(1 + m^2/(-k^2 - i0)) + (m^4/k^2) \ln(-k^2 - i0)/\mu^2 \Big\} . \qquad (45)
\end{aligned}
$$

If we spoil the tadpole by $\ln(-k^2 - i0)/\mu^2$, rather than by $\ln(m^2 - k^2 - i0)/\mu^2$ we get a more simple result:

$$
R \int \frac{\mathrm{d}k}{m^2 - k^2 - i0} = m^4 \int \frac{\mathrm{d}k \ln(-k^2 - i0)/\mu^2}{(-k^2 - i0)(m^2 - k^2 - i0)^2} . \qquad (46)
$$

[4] This was done in ref. [12] where more general prescriptions were formulated for logarithmically divergent diagrams with simple topology of subdivergences.

Let us now consider the sun-set diagram shown in Fig. 1b. If we used coordinate space arguments and started from the incompletely renormalized diagram

$$R'\Pi_\Gamma = (1 + \Delta(12) + \Delta(23) + \Delta(31))\Pi_\Gamma$$

(where $\Delta(12)$,... are counterterms associated with three overlapping one-loop subgraphs) and wanted to extend this functional from the space with deleted origin, $x = 0$, to the whole space, we would observe that the one-loop counterterms still vanish once we have $x \neq 0$. Thus we do not have enough space to perform renormalization of the sun-set diagram (and other similar diagrams) in two steps because extension to all x requires simultaneous introduction of the overall counterterm as well as counterterms for subgraphs.

To overcome this complication and arrive at general prescriptions we used, in [6], a trick of ref. [14] based on Fourier transform with respect to the squares of masses which were treated as squares of two-dimensional vectors. A better solution of this problem is just to use momentum space prescriptions (34)–(36) that happen to be more flexible. We state that these very prescriptions (34)–(36) *are valid for arbitrary diagrams*. For example, in the case of the sun-set diagram, we have (for definiteness, we have chosen the first line to 'spoil by a logarithm' for renormalization of the whole graph)

$$R\, F_\Gamma = (1 - \tilde{D}_{q,m})(1 - M^1_{q,m})R'\, F^{1,1}_\Gamma - c_{23} R \int \mathrm{d}k \frac{\ln(m^2 - k^2 - i0)/\mu^2}{m^2 - k^2 - i0}. \tag{47}$$

Here $F^{1,1}_\Gamma$ differs from F_Γ by the additional factor $\ln(m_1^2 - p_1^2 - i0)/\mu^2$ in the Feynman integral (p_1 is the momentum of this line), the renormalized value for the tadpole (with a logarithm) corresponding to the reduced graph $\Gamma/\{23\}$ is given by

$$R \int \mathrm{d}k \frac{\ln(m^2 - k^2 - i0)/\mu^2}{m^2 - k^2 - i0} = \frac{1}{2} \int \mathrm{d}k \left(\frac{1}{2} m \frac{\partial}{\partial m} - 1 \right) (1 - M^1_m) \frac{\ln^2(m^2 - k^2 - i0)/\mu^2}{m^2 - k^2 - i0}, \tag{48}$$

with $m = m_1$, and the constant c_{23} is found from eq. (36), i.e. in our case it is proportional to $\tilde{D} R F_{23}$ (in fact it does not depend on the renormalization scheme — compare [6]). Finally, counterterms for the given graph spoiled by the logarithm (that contribute to $R'\, F^{1,1}_\Gamma$), are found from eq. (38).

4.2 VVA anomaly as an example

Since differential renormalization is strictly four-dimensional it looks preferable for application in each situation when dimensional renormalization [15] meets difficulties, for instance, in theories with chiral and super symmetries (see, e.g. [8] where the initial version of differential renormalization [1] was successfully applied in such cases, in particular for calculation of anomalies). It is well-known that within dimensional renormalization the origin of anomalies turns out to be an inconsistency in definition

of γ_5 in dimensional regularization, and strictly in four dimensions one does not have such problems at all. Thus it is natural to ask where do the anomalies come from. As it was demonstrated in [8], the anomalies within differential renormalization appear because a system of linear equations for finite counterterms to satisfy all Ward identities turns out to overdetermined.

To see that the presented version of differential renormalization is calculationally simple and well suited for dealing with chiral problems let us once again consider the calculation of the ABJ triangle anomaly [16] as an example. Let $T_{\alpha\beta\rho}(q,p) = S_{\alpha\beta\rho}(q,p) + S_{\beta\alpha\rho}(-q-p,p)$ be the sum of two triangle diagrams contributing to one-loop Green function of the axial current $J_{5\rho} = \bar{\psi}\gamma_\rho\gamma_5\psi$, and let p correspond to the axial current and $q, -q-p$ to other two external lines. In accordance with general prescriptions (namely eq. (15) translated into momentum space language), the differentially renormalized quantities are given by

$$R_j S_{\alpha\beta\rho}(q,p) = -i\int \frac{\mathrm{d}k}{(2\pi)^4}(\tilde{D}_{q,p,m} - 1/2)(1 - M^0_{q,p,m}) \ln(m^2 - p_j^2 - i0)/\mu^2$$
$$\times \mathrm{tr}\left\{\gamma_\rho\gamma_5 \frac{1}{m - \not{k}}\gamma_\alpha \frac{1}{m - \not{k} - \not{q}}\gamma_\beta \frac{1}{m - \not{k} + \not{p}}\right\}, \qquad (49)$$

where $m^2 - i0$ prescriptions are omitted for brevity. Here $j = AV, VV, VA$ and p_j can be chosen as the momentum of the corresponding line (between vector or axial vertices). According to remark in subsect. 2.2 one can use an appropriate linear combination $R = \sum_j \xi_j R_j$, and we will do this below.

Since we want to have Ward identities in both vector channels,

$$q^\alpha R T_{\alpha\beta\rho}(q,p) = (q_\beta + p_\beta) T_{\alpha\beta\rho}(q,p) = 0, \qquad (50)$$

we should introduce the logarithms symmetrically in AV and VA lines, namely, choose $\xi_{AV} = \xi_{VA}$ in the above linear combination.

To calculate $q^\alpha R_j T_{\alpha\beta\rho}(q,p)$ we use simple commutation relations (28) and introduce auxiliary analytic regularization (as in an example in subsect. 4.1). by $\ln(m^2 - p_j^2 - i0)/\mu^2 \to -\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mu^{2\lambda}}{(m^2 - p_j^2 - i0)^\lambda}$. Then we observe that the action of the operator $(\tilde{D}_{q,p,m} - 1/2)(1 - M^0_{q,p,m})$ reduces to calculation of the finite part of the Laurent expansion in λ (actually the pole part turns out to be zero):

$$q^\alpha R_j S_{\alpha\beta\rho}(q,p) = -i\int \frac{\mathrm{d}k}{(2\pi)^4}\frac{\mu^{2\lambda}}{(m^2 - p_j^2 - i0)^\lambda}$$
$$\times q^\alpha \mathrm{tr}\left\{\gamma_\rho\gamma_5 \frac{1}{m - \not{k}}\gamma_\alpha \frac{1}{m - \not{k} - \not{q}}\gamma_\beta \frac{1}{m - \not{k} + \not{p}}\right\}\Bigg|_{\lambda=0}. \qquad (51)$$

Now we apply momentum space version of (27),

$$\frac{1}{m - \not{p}}\,\not{q}\,\frac{1}{m - \not{p} - \not{q}} = \frac{1}{m - \not{p} - \not{q}} - \frac{1}{m - \not{p}}. \qquad (52)$$

From coordinate space considerations, one sees that the result should be polynomial of the second degree in q, p, m. Furthermore, well-known formulae for traces of products of gamma matrices show that one can put $m = 0$. Then one uses relation $\mathrm{tr}\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_5 = 4i\epsilon_{\mu\nu\alpha\beta}$, eqs. (41), (42),

$$\int \mathrm{d}^4k \frac{k^\nu}{(-k^2 - i0)^{\lambda_1}(-(q-k)^2 - i0)^{\lambda_2}} = i\pi^2 G^{(1)}(\lambda_1, \lambda_2)\frac{q^\nu}{(-q^2 - i0)^{\lambda_1+\lambda_2-2}}, \tag{53}$$

with

$$G^{(1)}(\lambda_1, \lambda_2) = \frac{\Gamma(\lambda_1 + \lambda_2 - 2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\frac{\Gamma(3-\lambda_1)\Gamma(2-\lambda_2)}{\Gamma(5-\lambda_1-\lambda_2)}, \tag{54}$$

and the value (43) as well as

$$\left\{G(1+\lambda, 1) - 2G^{(1)}(1+\lambda, 1)\right\}\Big|_{\lambda=0} = 1/2, \quad \left\{G^{(1)}(2, \lambda)\right\}\Big|_{\lambda=0} = 1/2. \tag{55}$$

The final result is

$$q^\alpha R_{AV} T_{\alpha\beta\rho} = q^\alpha R_{VA} T_{\alpha\beta\rho} = \frac{i}{8\pi^2}, \quad q^\alpha R_{VV} T_{\alpha\beta\rho} = \frac{i}{4\pi^2} \tag{56}$$

so that we have Ward identities in the vector channels (50) for $R = R_{AV} + R_{VA} - R_{VV}$.

Within this choice of renormalization one can perform calculation of $p^\rho R S_{\alpha\beta\rho}(q, p)$ using

$$\frac{1}{m - \not{p}}\not{q}\gamma_5\frac{1}{m - \not{p} - \not{q}} = \gamma_5\frac{1}{m - \not{p} - \not{q}} - \frac{1}{m - \not{p}}\gamma_5 - 2m\frac{1}{m - \not{p}}\gamma_5\frac{1}{m - \not{p} - \not{q}}, \tag{57}$$

instead of (52), and the same technique as before, in particular formulae (41), (42), (53), (54), (55), with the result

$$p^\rho R T_{\alpha\beta\rho} = 2miRT_{5\alpha\beta} - \frac{i}{2\pi^2}\epsilon_{\alpha\beta\mu\nu}p^\mu q^\nu, \tag{58}$$

where $T_{5\alpha\beta}$ is one-loop contribution to the Green function of the pseudovector current $i\bar{\psi}\gamma_\rho\gamma_5\psi$. This leads to the well-known value of the VVA anomaly.

5 Conclusion

The prescriptions of differential renormalization scheme presented above are applicable for arbitrary diagrams and look more simple as compared with previous versions. Let us stress that the main features of differential renormalization are pulling out certain differential operators and introducing a logarithmic dependence into the diagrams involved. One can define it either in coordinate or momentum space (although momentum space turns out to be more flexible in some situations).

Since the mechanism of proving Ward identities in QED at the diagrammatical level is very transparent, it was possible to do this automatically, to all orders. Of course, the diagrammatical realization of non-Abelian gauge symmetries is rather non-trivial. One may certainly hope that the presented version of differential renormalization can be useful in treating non-Abelian gauge symmetries combined with super and chiral symmetries strictly in four dimensions at least in lower orders of perturbation theory. Since the differential operators that are pulled out during renormalization have very simple commutation relations with multiplication by momenta, the problem here is to control the logarithms that are generated by renormalization.

Finally we note that the presented version of differential renormalization is naturally supplied with strictly four-dimensional methods of calculation of Feynman integrals, in particular integration by parts [17] which is a four-dimensional analogue of the method of integration by parts within dimensional regularization [18] and is itself based on the differential renormalization and its infrared analogue.

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